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The behavior of permeable multi-cracks in a piezoelectric material

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Abstract

In this paper, the behavior of four parallel symmetric cracks in a piezoelectric material under anti-plane shear loading is studied by the Schmidt method for the permeable crack surface boundary conditions. By use of the Fourier transform, the problem can be solved with the help of two pairs of triple integral equations that the unknown variables are the jumps of the displacement across the crack surfaces. These equations are solved by means of the Schmidt method. The results show that the stress and the electric displacement intensity factors of cracks depend on the geometry of the crack. Contrary to the impermeable crack surface condition solution, it is found that the electric displacement intensity factors for the permeable crack surface conditions are much smaller than the results for the impermeable crack surface conditions.

Keywords: Piezoelectric materials; Schmidt method; The triple integral equations; Parallel cracks

1. Introduction

It is well known that piezoelectric materials produce an electric field when deformed and undergo deformation when subjected to an electric field. The coupling nature of piezoelectric materials has attracted wide applications in electric-mechanical and electric devices, such as electric-mechanical actuators, sensors and structures. When subjected to mechanical and electrical loads in service, these piezoelectric materials can fail prematurely due to defects, e.g., cracks, holes, etc. arising during their manufacture process. Therefore, it is of great importance to study the electro-elastic interaction and fracture behavior of piezoelectric materials, especially when multiple cracks are involved. In the past several years, theoretical studies of fracture in piezoelectric materials were carried out by many researchers (Deeg, 1980; Sosa and Pak, 1990; Suo et al., 1992; McMeeking, 1989; Zhang and Tong, 1996; Soh et al., 2000). It is interesting to note that very different results were obtained by changing the boundary conditions. To our knowledge, the

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electro-elastic behavior of four parallel symmetric permeable cracks under anti-plane shear loading in a piezoelectric material has not been studied.

In the present paper, the interaction between four parallel symmetrical cracks subjected to anti-plane shear loading in piezoelectric materials is investigated by use of the Schmidt method (Morse and Feshbach, 1958). It is a simple and convenient method for solving this problem. The Fourier transform technology is applied and a mixed boundary-value problem is reduced to two pairs of triple integral equations. In solving the triple integral equations, the jumps of the displacement across the crack surfaces are expanded in a series of Jacobi polynomials. This process is quite different from that adopted in previous works (Deeg, 1980; Sosa and Pak, 1990; Suo et al., 1992; McMeeking, 1989; Zhang and Tong, 1996; Soh et al., 2000). The form of solution is easy to understand. Numerical examples are provided to show the effect of the geometry of the cracks upon the stress intensity factor of the cracks.

2. Formulation of the problem

It is assumed that there are four parallel symmetric cracks of length $(1-b)$ in a piezoelectric material as shown in Fig. 1. h is the distance between cracks (The solution of four parallel symmetric cracks of length $a-b$ in the piezoelectric materials can easily be obtained by a simple change in the numerical values of the present paper, $a>b>0$.). The piezoelectric boundary-value problem for anti-plane shear is simplified considerably if we consider only the out-of-plane displacement and the in-plane electric fields. As discussed in Soh et al. (2000) work, since no opening displacement exists for the present anti-plane problem, the crack surfaces can be assumed to be in perfect contact. Accordingly, the permeable condition will be enforced in the present study, i.e., both the electric potential and the normal electric displacement are assumed to be continuous across the crack surfaces. So the boundary conditions of the present problem are:

$$w^{(1)}=w^{(2)},\quad \tau_{yz}^{(1)}=\tau_{yz}^{(2)},\quad \phi^{(1)}=\phi^{(2)},\quad D_y^{(1)}=D_y^{(2)},\ y=h,\quad b>|x|\geqslant 0,\ |x|>1 \tag{1}$$

$$w^{(2)}=w^{(3)},\quad \tau_{yz}^{(2)}=\tau_{yz}^{(3)},\quad \phi^{(2)}=\phi^{(3)},\quad D_y^{(2)}=D_y^{(3)},\ y=0,\quad b>|x|\geqslant 0,\ |x|>1 \tag{2}$$

$$\tau_{yz}^{(1)}=\tau_{yz}^{(2)}=-\tau_0,\quad \phi^{(1)}=\phi^{(2)},\quad D_y^{(1)}=D_y^{(2)},\ y=h,\quad b\leqslant|x|\leqslant 1 \tag{3}$$

$$\tau_{yz}^{(2)}=\tau_{yz}^{(3)}=-\tau_0,\quad \phi^{(2)}=\phi^{(3)},\quad D_y^{(2)}=D_y^{(3)},\ y=0,\quad b\leqslant|x|\leqslant 1 \tag{4}$$

$$w^{(1)}=w^{(2)}=w^{(3)}=0\quad \text{for } (x^2+y^2)^{1/2}\to\infty \tag{5}$$

where τ_{zk}, D_k $(k=x,y)$ are the anti-plane shear stress and in-plane electric displacement, respectively. w and ϕ are the mechanical displacement and the electric potential. Note that all quantities with superscript k

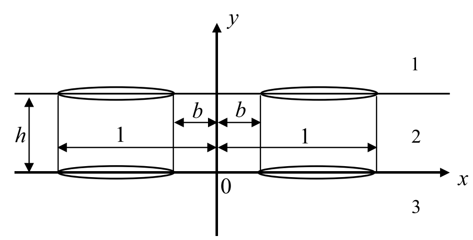

Fig. 1. Four parallel symmetric cracks in a piezoelectric material.

($k = 1, 2, 3$) refer to the upper half plane 1, the layer 2 and the lower half plane 3 as shown in Fig. 1. In this paper, we only consider that τ_0 is positive.

The constitutive equations can be written as

$$\tau_{zk} = c_{44} w_{,k} + e_{15}\phi_{,k}, \quad D_k = e_{15} w_{,k} - \varepsilon_{11}\phi_{,k} \tag{6}$$

c_{44}, e_{15} and ε_{11} are the shear modulus, piezoelectric coefficient and dielectric parameter. The anti-plane governing equations are

$$c_{44}\nabla^2 w + e_{15}\nabla^2\phi = 0, \quad e_{15}\nabla^2 w - \varepsilon_{11}\nabla^2\phi = 0 \tag{7}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the two-dimensional Laplace operator. Because of the assumed symmetry in geometry and loading, it is sufficient to consider the problem for $0 \leqslant x < \infty$, $0 \leqslant y < \infty$ only. A Fourier transform is applied to Eqs. (6) and (7). Assume that the solution is

$$\begin{cases} w^{(1)}(x,y) = \dfrac{2}{\pi}\displaystyle\int_0^\infty A_1(s)\,\mathrm{e}^{-sy}\cos(sx)\,\mathrm{d}s \\ \phi^{(1)}(x,y) = \dfrac{e_{15}}{\varepsilon_{11}} w^{(1)}(x,y) + \dfrac{2}{\pi}\displaystyle\int_0^\infty B_1(s)\,\mathrm{e}^{-sy}\cos(sx)\,\mathrm{d}s \end{cases} \quad (y \geqslant h) \tag{8}$$

$$\begin{cases} w^{(2)}(x,y) = \dfrac{2}{\pi}\displaystyle\int_0^\infty [A_2(s)\,\mathrm{e}^{-sy} + B_2(s)\,\mathrm{e}^{sy}]\cos(sx)\,\mathrm{d}s \\ \phi^{(2)}(x,y) = \dfrac{e_{15}}{\varepsilon_{11}} w^{(2)}(x,y) + \dfrac{2}{\pi}\displaystyle\int_0^\infty [C_2(s)\,\mathrm{e}^{-sy} + D_2(s)\,\mathrm{e}^{sy}]\cos(sx)\,\mathrm{d}s \end{cases} \quad (h \geqslant y \geqslant 0) \tag{9}$$

$$\begin{cases} w^{(3)}(x,y) = \dfrac{2}{\pi}\displaystyle\int_0^\infty A_3(s)\,\mathrm{e}^{sy}\cos(sx)\,\mathrm{d}s \\ \phi^{(3)}(x,y) = \dfrac{e_{15}}{\varepsilon_{11}} w^{(3)}(x,y) + \dfrac{2}{\pi}\displaystyle\int_0^\infty B_3(s)\,\mathrm{e}^{sy}\cos(sx)\,\mathrm{d}s \end{cases} \quad (y \leqslant 0) \tag{10}$$

where $\mu = c_{44} + (e_{15}^2/\varepsilon_{11})$, $A_1(s)$, $B_1(s)$, $A_2(s)$, $B_2(s)$, $C_2(s)$, $D_2(s)$, $A_3(s)$ and $B_3(s)$ are unknown functions. The gap functions of the crack surface displacements and the electric potentials are defined as follows:

$$f_1(x) = w^{(1)}(x,h^+) - w^{(2)}(x,h^-), \quad f_{\phi 1}(x) = \phi^{(1)}(x,h^+) - \phi^{(2)}(x,h^-) \tag{11}$$

$$f_2(x) = w^{(2)}(x,0^+) - w^{(3)}(x,0^-), \quad f_{\phi 2}(x) = \phi^{(2)}(x,0^+) - \phi^{(3)}(x,0^-) \tag{12}$$

Substituting Eqs. (8)–(10) into Eqs. (11) and (12), applying the Fourier transform and the boundary conditions, it can be obtained

$$\bar{f}_1(s) = [A_1(s) - A_2(s)]\mathrm{e}^{-sh} - B_2(s)\,\mathrm{e}^{sh} \tag{13}$$

$$\bar{f}_{\phi 1}(s) = \frac{e_{15}}{\varepsilon_{11}}\bar{f}_1(s) + [B_1(s) - C_2(s)]\mathrm{e}^{-sh} - D_2(s)\,\mathrm{e}^{sh} = 0 \tag{14}$$

$$\bar{f}_2(s) = A_2(s) + B_2(s) - A_3(s), \quad \bar{f}_{\phi 2}(s) = \frac{e_{15}}{\varepsilon_{11}}\bar{f}_2(s) + C_2(s) + D_2(s) - B_3(s) = 0 \tag{15}$$

Here, a superposed bar indicates the Fourier transform throughout the paper.

Substituting Eqs. (8)–(10) into Eqs. (6), applying the Fourier transform and the boundary conditions, it can be obtained

$$\mu A_1(s)\,\mathrm{e}^{-sh} + e_{15}B_1(s)\,\mathrm{e}^{-sh} = \mu[A_2(s)\,\mathrm{e}^{-sh} - B_2(s)\,\mathrm{e}^{sh}] + e_{15}[C_2(s)\,\mathrm{e}^{-sh} - D_2(s)\,\mathrm{e}^{sh}] \tag{16}$$

$$[B_1(s) - C_2(s)]\mathrm{e}^{-2sh} + D_2(s) = 0 \tag{17}$$

$$\mu[A_2(s) - B_2(s)] + e_{15}[C_2(s) - D_2(s)] = -\mu A_3(s) - e_{15}B_3(s), \quad C_2(s) - D_2(s) + B_3(s) = 0 \tag{18}$$

By solving eight Eqs. (13)–(18) with eight unknown functions $A_1(s)$, $B_1(s)$, $A_2(s)$, $B_2(s)$, $C_2(s)$, $D_2(s)$, $A_3(s)$, $B_3(s)$ and applying the boundary conditions (3) and (4), it can be obtained:

$$\int_0^\infty sc_{44}[\bar{f}_1(s) + e^{-sh}\bar{f}_2(s)]\cos(sx)\,\mathrm{d}s = \pi\tau_0, \quad b \leqslant x \leqslant 1 \tag{19}$$

$$\int_0^\infty sc_{44}[\bar{f}_2(s) + e^{-sh}\bar{f}_1(s)]\cos(sx)\,\mathrm{d}s = \pi\tau_0, \quad b \leqslant x \leqslant 1 \tag{20}$$

$$\int_0^\infty \bar{f}_1(s)\cos(sx)\,\mathrm{d}s = 0, \quad \int_0^\infty \bar{f}_2(s)\cos(sx)\,\mathrm{d}s = 0 \qquad b > x \geqslant 0, \ x > 1 \tag{21}$$

From Eqs. (19)–(21), it can be obtained

$$\bar{f}_1(s) = \bar{f}_2(s) \Rightarrow f_1(x) = f_2(x), \quad \tau_{yz}^{(1)}(x,h) = \tau_{yz}^{(2)}(x,h) = \tau_{yz}^{(2)}(x,0) = \tau_{yz}^{(3)}(x,0) \tag{22}$$

$$D_y^{(1)}(x,h) = D_y^{(2)}(x,h) = D_y^{(2)}(x,0) = D_y^{(3)}(x,0) \tag{23}$$

To determine the unknown functions $\bar{f}_1(s)$ and $\bar{f}_2(s)$, the triple integral equations (19)–(21) must be solved.

3. Solution of the triple integral equation

The Schmidt method (Morse and Feshbach, 1958) is used to solve the triple integral equations (19)–(21). The gap functions of the crack surface displacement are represented by the following series:

$$f_1(x) = \sum_{n=0}^{\infty} a_n P_n^{(1/2,1/2)}\left(\frac{x - \frac{1+b}{2}}{\frac{1-b}{2}}\right)\left(1 - \frac{\left(x - \frac{1+b}{2}\right)^2}{\left(\frac{1-b}{2}\right)^2}\right)^{1/2}, \quad \text{for } b \leqslant x \leqslant 1, \ y = 0 \tag{24}$$

where a_n are unknown coefficients to be determined and $P_n^{(1/2,1/2)}(x)$ are Jacobi polynomials (Gradshteyn and Ryzhik, 1980). The Fourier transformation of Eq. (24) is (Erdelyi, 1954)

$$\bar{f}_1(s) = \sum_{n=0}^{\infty} a_n Q_n G_n(s)\frac{1}{s}J_{n+1}\left(s\frac{1-b}{2}\right) \tag{25}$$

$$Q_n = 2\sqrt{\pi}\frac{\Gamma\left(n+1+\frac{1}{2}\right)}{n!}, \quad G_n(s) = \begin{cases} (-1)^{n/2}\cos\left(s\dfrac{1+b}{2}\right), & n = 0,2,4,6,\ldots \\ (-1)^{(n+1)/2}\sin\left(s\dfrac{1+b}{2}\right), & n = 1,3,5,7,\ldots \end{cases}$$

where $\Gamma(x)$ and $J_n(x)$ are the Gamma and Bessel functions, respectively.

Substituting Eq. (25) into Eqs. (19)–(21), Eq. (21) have been automatically satisfied. Then Eqs. (19) and (20) reduce to the form after integration with respect to x for $b \leqslant x \leqslant 1$,

$$c_{44}\sum_{n=0}^{\infty} a_n Q_n \int_0^\infty s^{-1}[1 + \mathrm{e}^{-sh}]G_n(s)J_{n+1}\left(s\frac{1-b}{2}\right)[\sin(sx) - \sin(sb)]\,\mathrm{d}s = \pi\tau_0(x - b) \tag{26}$$

As discussed in Itou (1978) and Zhou and Shen (1999), Eq. (26) can now be solved for the coefficients a_n by the Schmidt's method (Morse and Feshbach, 1958).

4. Intensity factors

The entire stress field and the electric displacement can be obtained as the coefficients a_n are known. However, in fracture mechanics, it is of importance to determine the stress τ_{yz} and the electric displacement D_y in the vicinity of the crack tips. $\tau_{yz}^{(1)}$, $\tau_{yz}^{(2)}$, $\tau_{yz}^{(3)}$, $D_y^{(1)}$, $D_y^{(2)}$ and $D_y^{(3)}$ along the crack line can be expressed respectively as

$$\tau_{yz}^{(1)}(x,h) = \tau_{yz}^{(2)}(x,h) = \tau_{yz}^{(2)}(x,0) = \tau_{yz}^{(3)}(x,0) = \tau_{yz} = -\frac{c_{44}}{\pi}\sum_{n=0}^{\infty} a_n Q_n \int_0^{\infty} [1+\mathrm{e}^{-sh}] G_n(s) J_{n+1}(sl)\cos(xs)\,\mathrm{d}s \tag{27}$$

$$D_y^{(1)}(x,h) = D_y^{(2)}(x,h) = D_y^{(2)}(x,0) = D_y^{(3)}(x,0) = D_y = -\frac{e_{15}}{\pi}\sum_{n=0}^{\infty} a_n Q_n \int_0^{\infty} [1+\mathrm{e}^{-sh}] G_n(s) J_{n+1}(sl)\cos(xs)\,\mathrm{d}s \tag{28}$$

An examination of Eqs. (27) and (28), the singular portions of the stress field and the electric displacement can be expressed respectively as following

$$\tau = -\frac{c_{44}}{2\pi}\sum_{n=0}^{\infty} a_n Q_n H_n(b,x), \quad D = -\frac{e_{15}}{2\pi}\sum_{n=0}^{\infty} a_n Q_n H_n(b,x) \tag{29}$$

where

$$H_n(b,x) = (-)^{n+1} F_1(b,x,n), \quad n = 0,1,2,3,4,5,\ldots \quad (\text{for } 0 < x < b)$$

$$H_n(b,x) = -F_2(b,x,n), \quad n = 0,1,2,3,4,5,\ldots \quad (\text{for } 1 < x)$$

$$F_1(b,x,n) = \frac{2(1-b)^{n+1}}{\sqrt{(1+b-2x)^2-(1-b)^2}\,[1+b-2x+\sqrt{(1+b-2x)^2-(1-b)^2}]^{n+1}}$$

$$F_2(b,x,n) = \frac{2(1-b)^{n+1}}{\sqrt{(2x-1-b)^2-(1-b)^2}\,[2x-1-b+\sqrt{(2x-1-b)^2-(1-b)^2}]^{n+1}}$$

At the left tip of the right crack, we obtain the stress intensity factor K_{L} as

$$K_{\mathrm{L}} = \lim_{x\to b^-}\sqrt{2\pi(b-x)}\cdot\tau = c_{44}\sqrt{\frac{1}{2\pi(1-b)}}\sum_{n=0}^{\infty}(-1)^n a_n Q_n \tag{30}$$

At the right tip of the right crack, we obtain the stress intensity factor K_{R} as

$$K_{\mathrm{R}} = \lim_{x\to 1^+}\sqrt{2\pi(x-1)}\cdot\tau = c_{44}\sqrt{\frac{1}{2\pi(1-b)}}\sum_{n=0}^{\infty} a_n Q_n \tag{31}$$

At the left tip of the right crack, we obtain the electric displacement intensity factor $K_{\rm L}^{\rm D}$ as

$$K_{\rm L}^{\rm D} = \lim_{x \to b^-} \sqrt{2\pi(b-x)} \cdot D = e_{15}\sqrt{\frac{1}{2\pi(1-b)}} \sum_{n=0}^{\infty} (-1)^n a_n Q_n = \frac{e_{15}}{c_{44}} K_{\rm L} \tag{32}$$

At the right tip of the right crack, we obtain the electric displacement intensity factor $K_{\rm R}^{\rm D}$ as

$$K_{\rm R}^{\rm D} = \lim_{x \to 1^+} \sqrt{2\pi(x-1)} \cdot D = e_{15}\sqrt{\frac{1}{2\pi(1-b)}} \sum_{n=0}^{\infty} a_n Q_n = \frac{e_{15}}{c_{44}} K_{\rm R} \tag{33}$$

5. Numerical calculations and discussion

From the works Itou (1978) and Zhou and Shen (1999), it can be seen that the Schmidt method is performed satisfactorily if the first ten terms of the infinite series (26) are obtained. So the stress intensity factor K and the electric displacement intensity factor $D_{\rm L}$ can be calculated numerically. In the computations, the piezoelectric material is assumed to be the commercially available piezoelectric PZT-4. The material constants of PZT-4 are $c_{44} = 2.56$ ($\times 10^{10}$ N/m^2), $e_{15} = 12.7$ (c/m^2) and $\varepsilon_{11} = 64.6$ ($\times 10^{-10}$ c/Vm^2). The results of the present paper are shown in Figs. 2–7. From the results, the following observations are very significant:

(i) The stress and the electric displacement intensity factors depend on the crack length and the distance between four parallel cracks. (ii) The stress and the electric displacement intensity factors of the four parallel cracks decrease as the distance between the parallel cracks decreases. The stress intensity factors and the electric displacement intensity factors of the four parallel cracks increase when the length of cracks increases. This phenomenon is called crack shielding effect as discussed in Ratwani's paper (Ratwani and Gupta, 1974). (iii) The electric displacement intensity factors for the permeable crack surface conditions are much smaller than the results for the impermeable crack surface conditions as shown in Zhou's paper (Zhou and Shen, 1999). (iv) The stress intensity factor does not depend on the material constants. However, the electric displacement intensity factor depends on the shear modulus and the dielectric parameter as shown in Eqs. (32) and (33). (v) The stress and the electric displacement intensity factors at the inner crack tips are larger than ones at the outer crack tips.

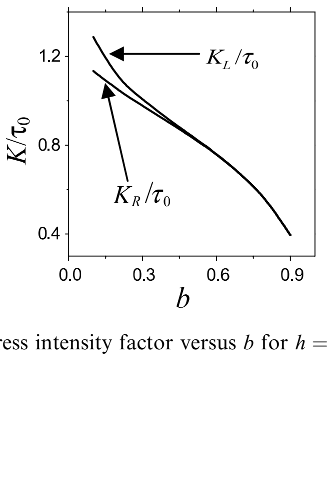

Fig. 2. The stress intensity factor versus b for $h = 0.5$ (PZT-4).

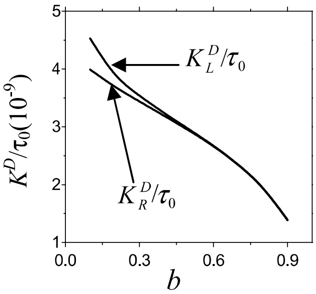

Fig. 3. The electric displacement intensity factor versus b for $h = 0.5$ (PZT-4).

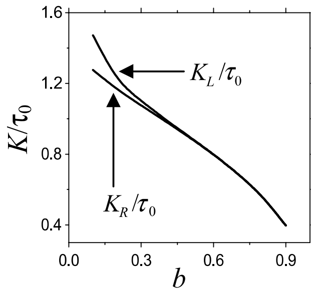

Fig. 4. The stress intensity factor versus b for $h = 4.0$ (PZT-4).

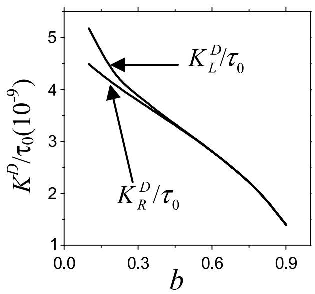

Fig. 5. The electric displacement intensity factor versus b for $h = 4.0$ (PZT-4).

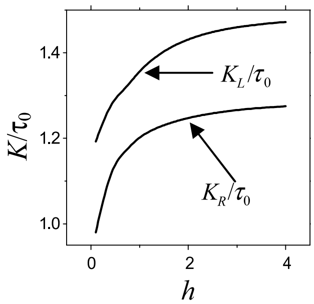

Fig. 6. The stress intensity factor versus h for $b = 0.1$ (PZT-4).

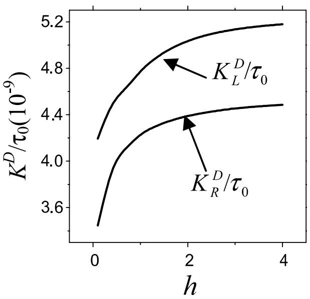

Fig. 7. The electric displacement intensity factor versus h for $b = 0.1$ (PZT-4).

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